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point.[1] In terms of our former analogy we may say that if we have a set of "cross-section" values of a function (which in our application would be a set of values for a constant t) and of its first derivative, and know that the function is characterized and specialized by the fact that it satisfies some very general condition (which condition while very general must get at the very root of the matter) we can completely determine the function at all other points. This general condition, in the case of ordinary potential and many allied problems is, of course, given by the ordinary Laplacian equation $\Delta^2 V = 0$. This suggests that if, by careful study, we could but obtain some universally applicable and fundamentally far-reaching principle—some truly dynamic theory of history, to use Henry Adams's phrase, we could probably thus cut down enormously upon the amount of contemporary data necessary.

This analogy may perhaps all be interpreted as merely strengthening Wells's point that in science effect seems inevitably to follow cause in an unending and inescapable chain of events. There is perhaps no more striking example of the adequacy of causation than that contained in the fact that the value and behavior of y a million million miles down the x axis is inevitably determined by the way in which the curve crosses the y axis; and it is a natural and, it would seem, not ridiculous hope to expect that some day there will be an historian bold enough and wise enough to write down a few of the equations of history which will hold not only for the minus sign, but also for the plus.

II. A Proof of the Definite Integral Formula.

By Wayne Sensenig, Conshohocken, Pa.

On pages 214–216 of Lamb's *Infinitesimal Calculus* several special cases of the definite integral are calculated "ab initio." [2] The following direct proof of the formula for the definite integral covers cases where y can be expressed as a *convergent* power series in x over the *closed* interval considered in the definite integral. The area under a curve is used as an illustration, but the form of proof is obviously independent of any special geometric interpretation. Use is made of the following well-known formula:[3]

$$\begin{aligned} S_m &= 1^m + 2^m + \cdots + n^m \\ &= \frac{n^{m+1}}{m+1} + \tfrac{1}{2}n^m + B_1\frac{m}{2!}n^{m-1} - B_3\frac{m(m-1)(m-2)}{4!}n^{m-3} \\ &\quad + B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6!}n^{m-5} - \cdots, \end{aligned}$$

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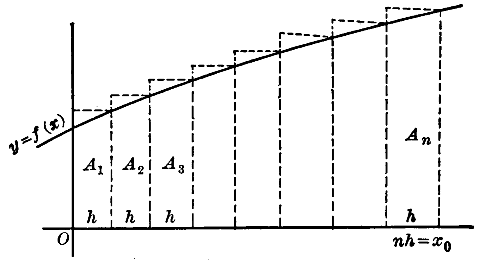
[1] If it is desired to calculate at points exterior to S the regularity of V at infinity is also necessary. The above mathematical fact does not correspond perfectly to the previous example, but the aspect of the logical content with which we are concerned is closely enough connected with the former to serve our purpose.

[2] Similar exercises are found in various text-books. Problems of the same character have from time to time appeared in the Monthly.—Editor.

[3] Hall and Knight, *Higher Algebra*, p. 337.

ending with a term in n or n^2; the B's being the Bernoulli numbers:

$$B_1 = \tfrac{1}{6},\quad B_3 = \tfrac{1}{30},\quad B_5 = \tfrac{1}{42},\quad B_7 = \tfrac{1}{30},\quad B_9 = \tfrac{5}{66},\quad \cdots.$$



Let the interval of integration be $0 \leqq x \leqq x_0$, and suppose

$$y = f(x) = a_0 + a_1x + a_2x^2 + \cdots,$$

the series converging in an interval extending beyond x_0. Then, since the convergence must be uniform in the closed interval $|x| \leqq x_0$, any given ϵ, however small, will determine a value of m, such that

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + R_m(x),$$

where

$$|R_m(x)| < \frac{\epsilon}{2x_0}, \qquad 0 \leqq x \leqq x_0.$$

Divide the interval into n equal parts each of length h, and form the Cauchy sum in the usual way. With reference to the accompanying figure, we have:

$$\begin{aligned}
A_1 &= a_0h + a_1h^2 + a_2h^3 + \cdots + a_mh^{m+1} + hR_m(h),\\
A_2 &= a_0h + 2a_1h^2 + 2^2a_2h^3 + \cdots + 2^ma_mh^{m+1} + hR_m(2h),\\
&\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\\
A_n &= a_0h + na_1h^2 + n^2a_2h^3 + \cdots + n^ma_mh^{m+1} + hR_m(nh).
\end{aligned}$$

Thus

$$A = A_1 + A_2 + \cdots + A_n = s_{m,\,n} + hr_{m,\,n},$$

where

$$s_{m,\,n} = a_0nh + a_1h^2S_1 + a_2h^3S_2 + \cdots + a_mh^{m+1}S_m,$$

$$r_{m,\,n} = h[R_m(h) + R_m(2h) + \cdots + R_m(nh)].$$

Evidently

$$|r_{m,\,n}| < hn\frac{\epsilon}{2x_0} = \frac{\epsilon}{2};$$

so that independently of n, $r_{m,\,n}$ can be made arbitrarily small.

Furthermore, we have

$$nh = x_0$$

$$\lim_{n\to\infty} h^2S_1 = \lim_{n\to\infty} h^2\left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{{x_0}^2}{2},$$

$$\lim_{n\to\infty} h^3S_2 = \lim_{n\to\infty} h^3\left(\frac{n^3}{3} + \frac{n^2}{2} + B_1\frac{2}{2!}n\right) = \frac{{x_0}^3}{3},$$

$$\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot$$

$$\lim_{n\to\infty} h^{m+1}S_m = \lim_{h\to\infty} h^{m+1}\left(\frac{n^{m+1}}{m+1} + \frac{n^m}{2} + \cdots\right) = \frac{{x_0}^{m+1}}{m+1};$$

and therefore

$$\lim_{n\to\infty} s_{m,\,n} = a_0x_0 + a_1\frac{x_0^2}{2} + \cdots + a_m\frac{x_0^{m+1}}{m+1}.$$

The last expression is the sum of $(m + 1)$ terms of the series

$$a_0x_0 + a_1\frac{x_0^2}{2} + a_2\frac{x_0^3}{3} + \cdots.$$

If we choose n so great that $s_{m,\,n}$ differs from its limit by less than $\epsilon/2$, then since $|r_{m,\,n}| < \epsilon/2$ for all n, we see that A differs from the sum of $(m + 1)$ terms of the above series by less than ϵ. Since the series is readily shown to converge, it follows that the limit of A, as n becomes infinite, exists and is equal to the value of the series:

$$\int_0^{x_0} f(x)dx = a_0x_0 + a_1\frac{x_0^2}{2} + a_2\frac{x_0^3}{3} + \cdots.$$

The result can easily be extended to the case in which the lower limit of integration is not zero.

III. On a Formula of Plane Trigonometry.

By M. C. Baudin, Miami University.

The formulas for finding the angles of a triangle when the three sides are given, namely:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

where s is one half the perimeter of the triangle, may be derived trigonometrically from the law of sines in the following manner.

Consider the equations:

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{s-c}{s}, \tag{1}$$

$$\frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-b}{s}, \tag{2}$$

$$\frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-a}{s}. \tag{3}$$

Let us change the left-hand members of these equations into quotients of products. The numerator of (1) may be written

$$\sin A + \sin B - \sin C + \sin(A + B + C). \tag{4}$$

Now

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin (A + B + C) - \sin C = 2 \sin \frac{A+B}{2} \cos \left(C + \frac{A+B}{2} \right)$$

and (4) becomes

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} + \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \cos \frac{B+C}{2} \cos \frac{A+C}{2}$$

or

$$4 \cos \frac{C}{2} \sin \frac{A}{2} \sin \frac{B}{2}.$$

The denominator of (1) may be written

$$\sin A + \sin B + \sin C - \sin (A + B + C).$$

Changing $\sin A + \sin B$ and then $\sin C - \sin (A + B + C)$ into products and adding the results, we get

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} - \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{A+C}{2}$$

or

$$4 \cos \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

We therefore have

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2}.$$

In the same way we obtain

$$\frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{C}{2},$$

$$\frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

Substituting these values for the left-hand members in (1), (2) and (3), and solving those equations, we get

$$\tan \frac{A}{2} = \pm \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{B}{2} = \pm \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{C}{2} = \pm \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

The positive sign must evidently be chosen before the radical, since $A/2$, $B/2$, $C/2$ cannot be greater than $\pi/2$.